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ORIGINAL PAPER

Multidimensional Magic Mountains and Matrix Art for the generalized repeat space theory

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Abstract The Asymptotic Linearity Theorem (ALT), which proves the Fukui conjecture in a broader context, plays a significant role in the repeat space theory (RST), which is the central unifying theory in the First and the Second Generation Fukui Project. Proving the Asymptotic Linearity Theorem Extension Conjecture (ALTEC) is a fundamental problem in the repeat space theory. The present paper constructs a class of functions $\mathrm{MagicMt}_\theta$, which serves as a powerful tool for proving the Asymptotic Linearity Theorem Extension Conjecture and related propositions. The d-dimensional generalization $\mu_{d,n,\theta}$ of $\mathrm{MagicMt}_\theta$, which is given in the present paper and is called a 'd-dimensional Magic Mountain', provides inwardly repeating fractals in multidimensional spaces useful for interdisciplinary research that uses the generalized repeat space theory.

Keywords The Fukui conjecture · Repeat space theory (RST) · Asymptotic Linearity Theorem Extension Conjecture (ALTEC) · Fractals in multidimensional spaces · Banach algebras

1 Introduction

In his later years, Kenichi Fukui (1918–1998, Nobel Prize 1981) presented several conjectures concerning the additivity problems of molecules having many identical moieties. Among them is the following which has been playing a significant role in the development of the repeat space theory (RST) (cf. [1–20]), which is the central

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unifying theory in the First (cf. [1,2]) and the Second (cf. [2,3]) Generation Fukui Project:

The Fukui Conjecture. *Let* $\{M_N\}$ *be a fixed element of the repeat space with block-size q, and let I be a fixed closed interval on the real line such that I contains all the eigenvalues of* M_N for *all positive integers N. Let* $\varphi_{1/2} : I \to \mathbb{R}$ *denote the function defined by* $\varphi_{1/2}(t) = |t|^{1/2}$*. Then, there exist real numbers* α *and* β *such that*

$$\mathrm{Tr}\varphi_{1/2}(M_N) = \alpha N + \beta + o(1) \tag{1.1}$$

as $N \to \infty$.

The Asymptotic Linearity Theorem (ALT), which proves the Fukui conjecture in a broader context, plays a significant role in the RST. Proving the Asymptotic Linearity Theorem Extension Conjecture (ALTEC) reproduced in Sect. 3 (cf. [20] for details) is thus a fundamental problem in the repeat space theory. The present paper constructs a class of functions $\mathrm{MagicMt}_\theta$, which serves as a powerful tool for proving the Asymptotic Linearity Theorem Extension Conjecture and related propositions. The d-dimensional generalization $\mu_{d,n,\theta}$ of $\mathrm{MagicMt}_\theta$, which is given in the present paper and is called a '***d*-dimensional Magic Mountain**', provides fractals in multidimensional spaces useful for interdisciplinary research that uses the generalized repeat space theory. Theory of Banach spaces and Banach algebras (cf. e.g. [24,25]) are fundamental for constructing the notion of the class of functions $\mathrm{MagicMt}_\theta$ and their d-dimensional generalization $\mu_{d,n,\theta}$.

2 Preparations

Throughout, let $\mathbb{Z}^+$, $\mathbb{Z}_0^+$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote respectively the set of all positive integers, nonnegative integers, integers, real numbers, and complex numbers. We also need the following symbols for our purpose of constructing the above-mentioned class of functions $\mathrm{MagicMt}_\theta$ and $\mu_{d,n,\theta}$.

Let A be a nonempty set, let $d \in \mathbb{Z}^+$, let

$$A^d := A \times A \times \cdots \times A, \tag{2.1}$$

where $\times$ denotes the Cartesian product repeated $d-1$ times. Let $J := [0, 1] \subset \mathbb{R}$, let $d \in \mathbb{Z}^+$, thus we have

$$J^d = J \times J \times \cdots \times J \subset \mathbb{R}^d.$$

Let ∂J^d denote the boundary of the set J^d in $\mathbb{R}^d$ with the usual Euclidean metric.

Let $C(J^d)$ denote the Banach space of all real-valued continuous functions on J^d equipped with the norm given by

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in J^d\}. \tag{2.2}$$

Let $C_0(J^d)$ denote the closed subspace of $C(J^d)$ defined by

$$C_0(J^d) = \{\varphi \in C(J^d) : \varphi(x) = 0 \text{ for all } x \in \partial J^d\}. \tag{2.3}$$

(**Note**: $C_0(J^d)$ is a Banach space since it is a closed subspace of the Banach space $C(J^d)$.) Let

$$C(\mathbb{R}^d) := \{\varphi : \varphi \text{ is a real-valued continuous function defined on } \mathbb{R}^d\}. \tag{2.4}$$

Let

$$\begin{aligned} C_0(\mathbb{R}^d) := \{\varphi \in C(\mathbb{R}^d) : \varphi(x) = 0 \text{ for all } x \in \partial J^d \text{ and} \\ \varphi(x) = \varphi(x+y) \text{ for all } x \in J^d \text{ and } y \in \mathbb{Z}^d\}. \end{aligned} \tag{2.5}$$

Let $^\wedge$ denote the linear operator $^\wedge$: $C_0(J^d) \to C_0(\mathbb{R}^d)$ that sends φ to the unique element $\hat{\varphi} \in C_0(\mathbb{R}^d)$ with $\hat{\varphi}\big|_{J^d} = \varphi$, where $\hat{\varphi}\big|_{J^d}$ denotes the restriction of the function $\hat{\varphi}$ to the set J^d. Note that the linear mapping $^\wedge$ is a bijection.

For each $n \in \mathbb{Z}^+$, define the linear operator $\hat{L}_n : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ by

$$\hat{L}_n(\varphi)(x) = \frac{1}{n}(\varphi(nx)). \tag{2.6}$$

For each $n \in \mathbb{Z}^+$, define the bounded linear operator $L_n : C_0(J^d) \to C_0(J^d)$ by

$$L_n(\varphi) = \hat{L}_n(\hat{\varphi})\big|_{J^d}. \tag{2.7}$$

Note that (2.6) and (2.7) imply that

$$\|L_n\| = \frac{1}{n}. \tag{2.8}$$

Let $\|\cdot\|_\infty$ denote the norm in $\mathbb{R}^d$ defined by

$$\|(x_1, x_2, \ldots, x_d)\|_\infty = \text{Max}\{|x_1|, |x_2|, \ldots, |x_d|\}. \tag{2.9}$$

Let

$$\gamma := \frac{1}{2}(1, 1, \ldots, 1) \in \mathbb{R}^d. \tag{2.10}$$

Let $\phi_0^d \in C_0(J^d)$ be the function defined by

$$\phi_0^d(x) = 1 - 2\|x - \gamma\|_\infty. \tag{2.11}$$

(**Note**: $\phi_0^d(x) = 0 \Leftrightarrow \|x - \gamma\|_\infty = \frac{1}{2} \Leftrightarrow x \in \partial J^d$.)

Let $\boldsymbol{B}(C_0(J^d))$ denote the Banach algebra of all bounded linear operators acting on $C_0(J^d)$. For each $d \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$ with $n \geq 2$, and $\theta \in \mathbb{R}$, define $\Lambda_{d,n,\theta} \in \boldsymbol{B}(C_0(J^d))$ by

$$\Lambda_{d,n,\theta} = \sum_{k=0}^{\infty} (\cos k\theta) L_n^k, \tag{2.12}$$

where L_n^0 denotes the identity operator and if $k \geq 1$, $L_n^k := L_n L_n \ldots L_n \in \boldsymbol{B}(C_0(J^d))$, and where L_n is composed with itself k times. Note that

$$\begin{aligned} \left\| \Lambda_{d,n,\theta} \right\| &\leq \sum_{k=0}^{\infty} \left\| (\cos k\theta) L_n^k \right\| \\ &\leq \sum_{k=0}^{\infty} \left\| L_n^k \right\| \leq \sum_{k=0}^{\infty} \| L_n \|^k = \frac{1}{1 - \frac{1}{n}}. \end{aligned} \tag{2.13}$$

3 Magic Mountains

We are now ready to define $\mu_{d,n,\theta} \in C_0(J^d)$, which is a d-dimensional generalization of the MagicMt$_\theta$. Let $d \in \mathbb{Z}^+$, let $n \in \mathbb{Z}^+$ with $n \geq 2$, let $\theta \in \mathbb{R}$, and let

$$\mu_{d,n,\theta} := \Lambda_{d,n,\theta} \left(\phi_0^d \right). \tag{3.1}$$

For each $\theta \in \mathbb{R}$, we can define MagicMt$_\theta$ in terms of $\mu_{d,n,\theta}$ given above:

$$\text{MagicMt}_\theta := \mu_{2,2,\theta}. \tag{3.2}$$

For each $k \in \mathbb{Z}_0^+$, define Pyramid$_k \in C_0(J^2)$ by

$$\text{Pyramid}_k = L_2^k \left(\phi_0^2 \right). \tag{3.3}$$

Then, we may also define MagicMt$_\theta$ in terms of Pyramid$_k$ given above:

$$\text{MagicMt}_\theta = \sum_{k=0}^{\infty} (\cos k\theta) \text{Pyramid}_k. \tag{3.4}$$

(**Note**: The latter way of defining MagicMt$_\theta$ by (3.4) is suitable when one constructs computer graphics of the graph of MagicMt$_\theta$, whereas the former way of defining MagicMt$_\theta$ is mathematically more powerful especially when one utilizes MagicMt$_\theta$ for the proof of the ALTEC [20]. See the ALTEC reproduced below. The detailed development along these lines together with the proof of the ALTEC will be published elsewhere.)

Now let us recall:

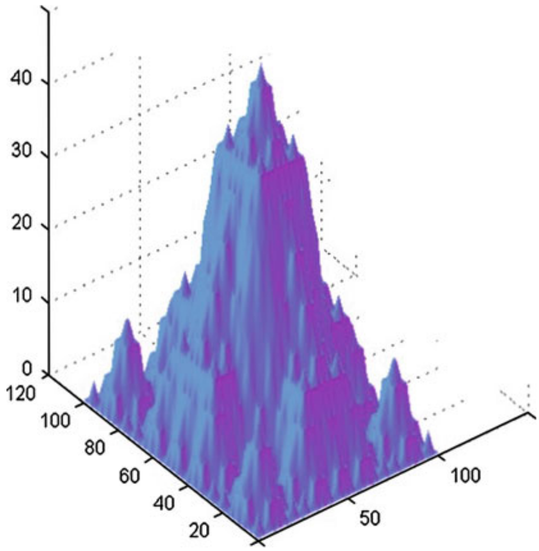


Fig. 1 Matrix Art picture of the graph of the function $\mathrm{MagicMt}_{\pi}$, nicknamed Tsuyama-castle function

Asymptotic Linearity Theorem Extension Conjecture (ALTEC $C(I)$ version) *The Asymptotic Linearity Theorem (ALT) cannot be extended from $AC(I)$ to $C(I)$, where $AC(I)$ denotes the functional space of all real valued absolutely continuous functions defined on the closed interval I, and $C(I)$ denotes the functional space of all real valued continuous functions defined on the closed interval I.*

This conjecture (ALTEC), which was referred to in Sect. 1 was first proved by the present author, and the second proof of this conjecture was recently obtained by him in a seminar called "Matrix Art Challenge Seminar" in Tsuyama National College of Technology (TNCT), in conjunction with operator $\Lambda_{d,n,\theta}$, the above defined continuous function with $\theta = \pi$: $\mathrm{MagicMt}_{\pi} : [0,\ 1] \times [0,\ 1] \rightarrow \mathbb{R}$, and the above Matrix Art picture (Fig. 1) of this function constructed by using MATLAB. The scale of the function has been changed in the pictures. The graph of the function $\mathrm{MagicMt}_{\pi}$ has an interesting self similarity and the nickname of the function $\mathrm{MagicMt}_{\pi}$ is '**Tsuyama-castle function**' ('Tsuyama-jyo kansu' in Japanese). Details of the applications of the operator $\Lambda_{d,n,\theta}$, the function $\mathrm{MagicMt}_{\pi}$, and the proofs of the ALTEC will be published elsewhere.

Remarks: The picture of the graph of $\mathrm{MagicMt}_{\pi}$ in the above Fig. 1 and the anaglyph picture of part of the graph of $\mathrm{MagicMt}_{\pi}$ given in Fig. 2 in Sect. 4 were first obtained in the Matrix Art Challenge Seminar in TNCT in parallel with the procedure of what is called the Niagara Project, which is a special new part of the on-going international, interdisciplinary, and inter-generational Second Generation Fukui Project.

4 Time evolution to $\mathrm{MagicMt}_{\pi}$

In this section, we provide the pictures of the time evolution of $\mathrm{MagicMt}_{\theta}$ with the following values of θ. The right side of each pair of pictures shows the contour map of

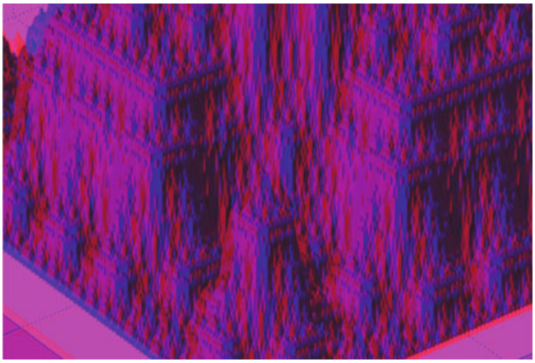
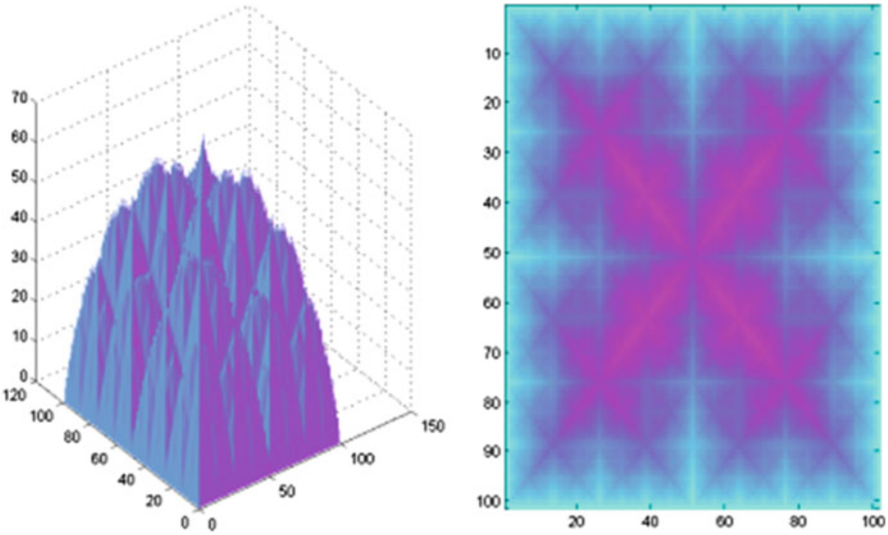


Fig. 2 Anaglyph Matrix Art of part of the graph of $\mathrm{MagicMt}_{\pi}$

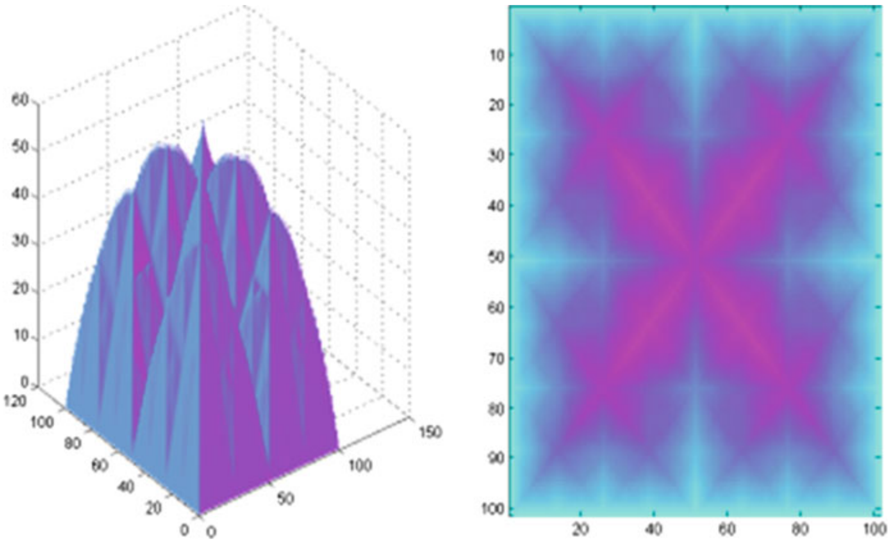
each function $\mathrm{MagicMt}_{\theta}$. (Here, the parameter $t := \frac{30}{\pi}\theta$ is considered as time ranging from 0 to 30 min.)

$$\theta=\frac{0\pi}{30}, \frac{3\pi}{30}, \frac{6\pi}{30}, \frac{9\pi}{30}, \frac{12\pi}{30}, \frac{15\pi}{30}, \frac{18\pi}{30}, \frac{21\pi}{30}, \frac{24\pi}{30}, \frac{27\pi}{30}, \frac{30\pi}{30}.$$

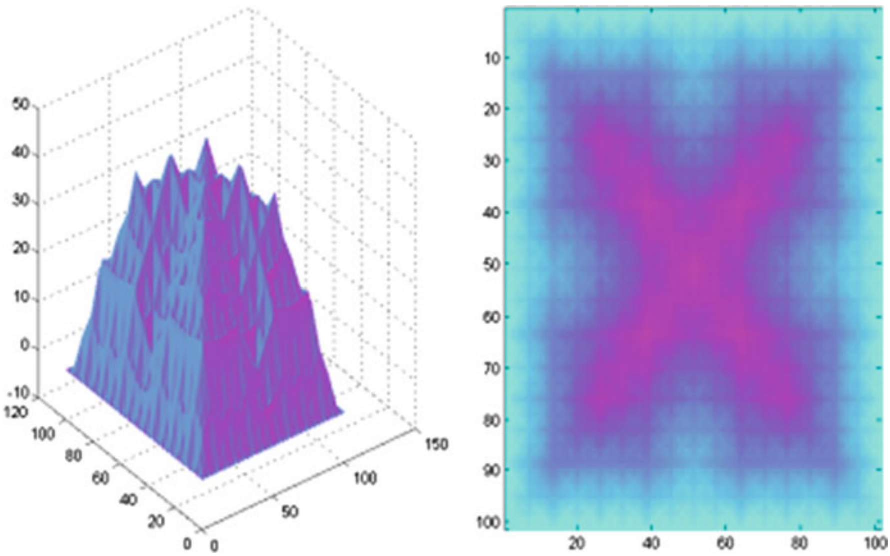
We remark that the scale of the functions has been changed in the pictures (1)–(11).



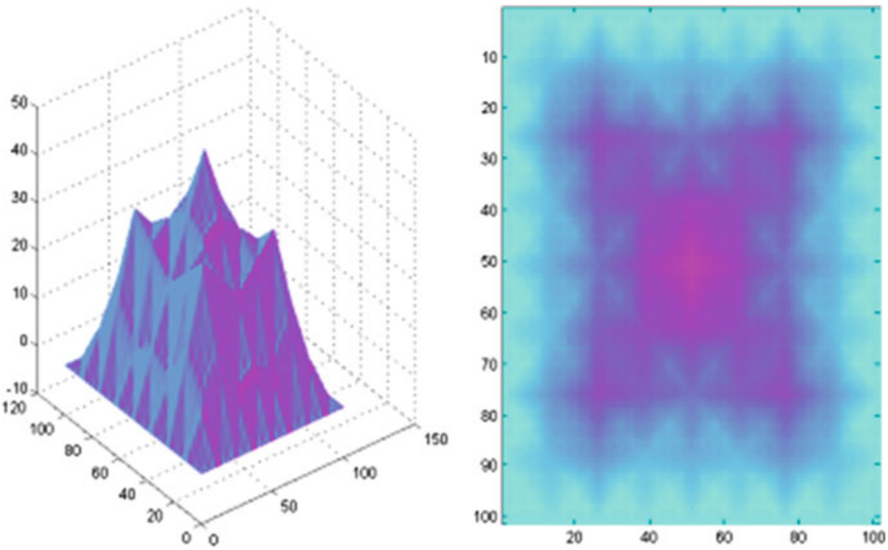
(1) $\theta=\frac{0\pi}{30}$



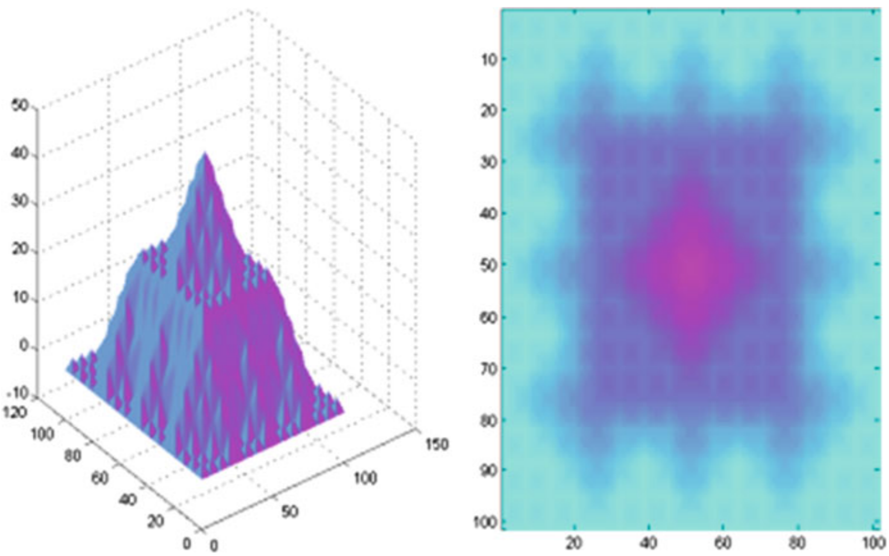
(2) $\theta = \frac{3\pi}{30}$



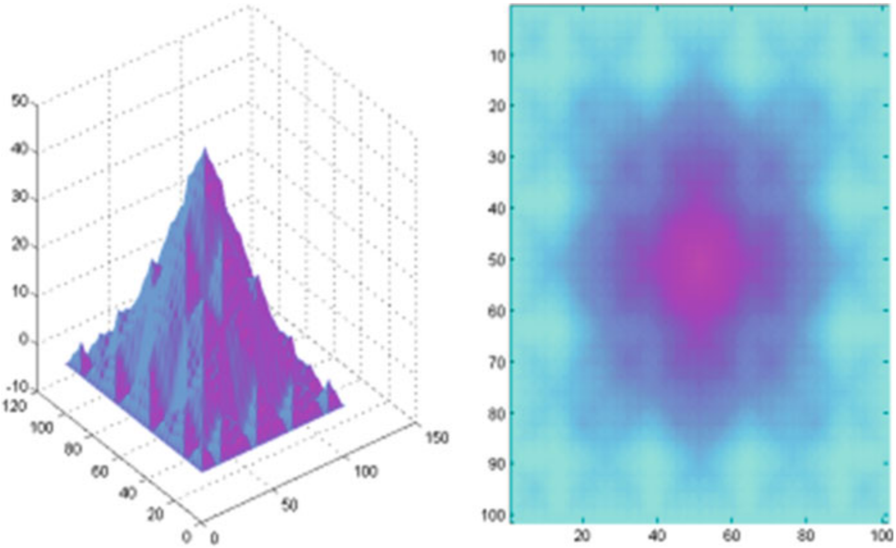
(3) $\theta = \frac{6\pi}{30}$



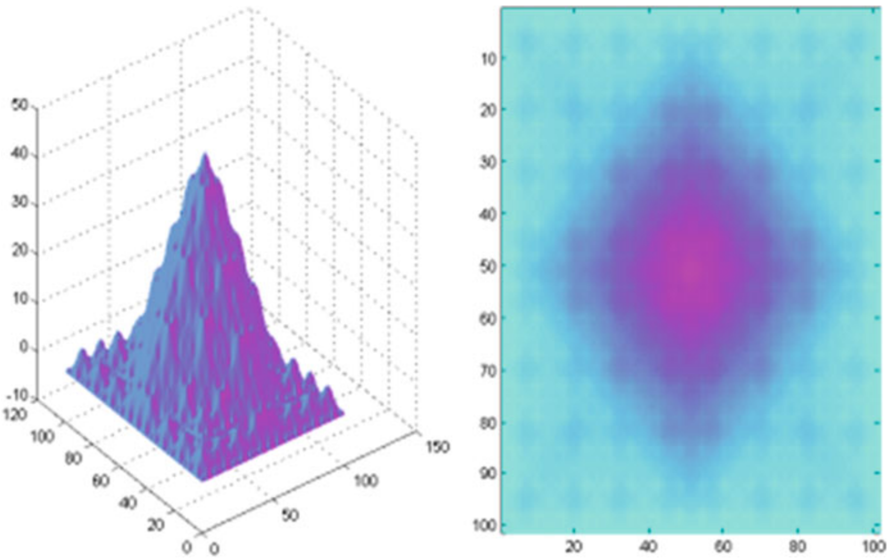
(4) $\theta = \frac{9\pi}{30}$



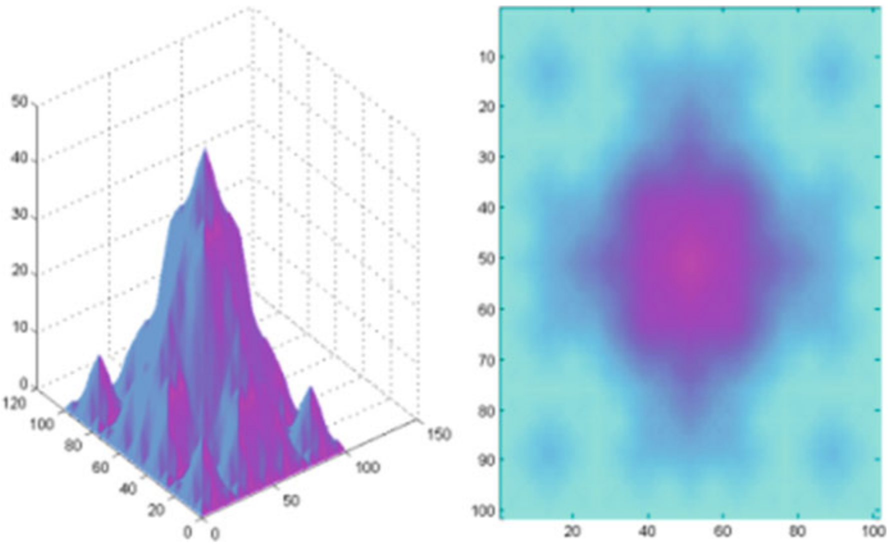
(5) $\theta = \frac{12\pi}{30}$



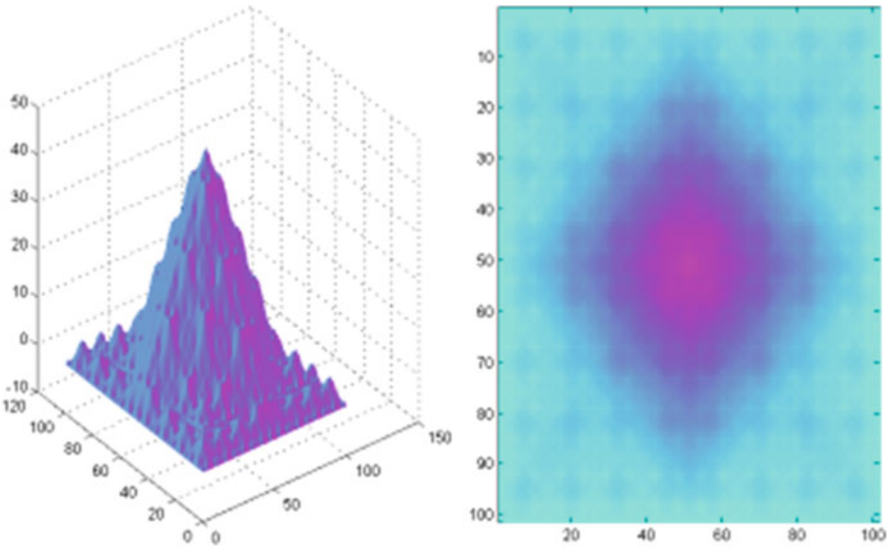
(6) $\theta = \frac{15\pi}{30}$



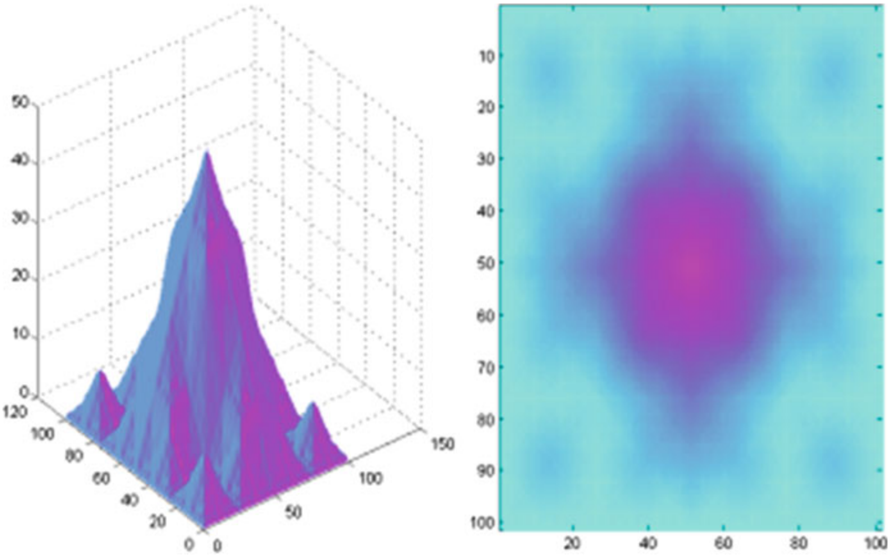
(7) $\theta = \frac{18\pi}{30}$



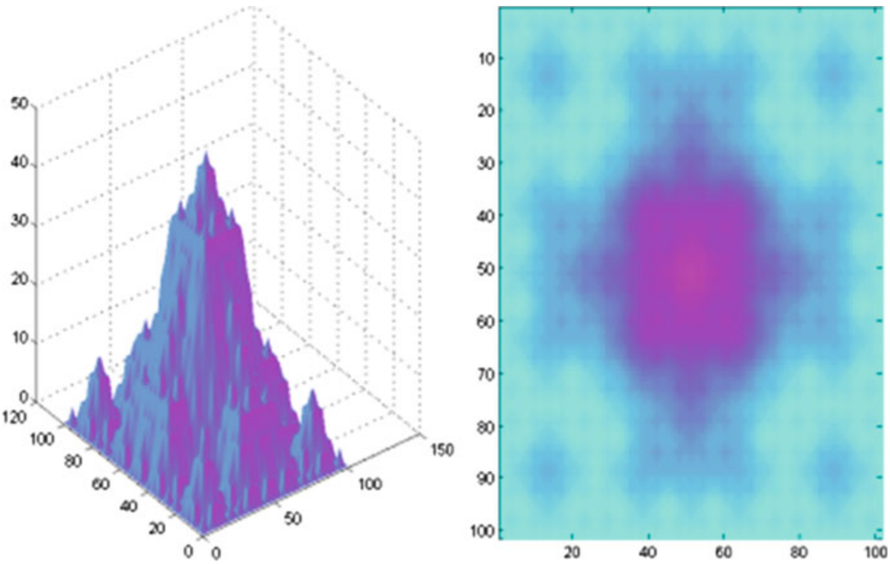
(8) $\theta = \frac{21\pi}{30}$



(9) $\theta = \frac{24\pi}{30}$



(10) $\theta = \frac{27\pi}{30}$



(11) $\theta = \frac{30\pi}{30}$

5 *d*-Dimensional Magic Mountains

It is noteworthy that the family of functions $\mu_{d,n,\theta} \in C_0(J^d)$ defined in Sect. 4 is an abundant source of fractals in multidimensional spaces, which are useful for interdisciplinary research that uses the generalized repeat space theory. Function $\mu_{d,n,\theta}$ is called a '***d*-dimensional Magic Mountain**'. For each $d \in \mathbb{Z}^+$, the nickname of the function $\mu_{d,2,\pi}$ is 'd-dimensional Tsuyama-castle function', or d-dimensional Tsuyama-jyo Kansu in Japanese. The function $(1/2)\mu_{1,2,0} : [0, 1] \rightarrow \mathbb{R}$ coincides with what is called the Takagi function [21], which is nowhere differentiable. Cf. [20–23] and references therein for this and related irregular functions. The reader is also invited to refer to Prof. H. Hironaka's public speech entitled 'Mathematics and the Sciences' [26] for an instructive account of the notion of self-similarity, fractal geometry, and of mathematical sciences.

In Fig. 2 in Sect. 4, we have provided Anaglyph Matrix Art of part of the graph of $\mathrm{MagicMt}_\pi$, which was created in the Matrix Art Challenge Seminar in Tsuyama National College of Technology. The red-blue glasses are needed to see the graph 3-dimentionally. (Cf. Ref. [8] and references therein for the origin and background of Matrix Art, Niagara Project, and the First Generation and the Second Generation Fukui Project.)

In Ref. [8], entitled 'Proof of the Fukui conjecture via resolution of singularities and related methods. V', theory of analytic (highly smooth) curves and resolution of singularities has been applied to prove the Fukui conjecture originating in chemistry. We remark that the investigations of highly smooth functions and of highly irregular functions are complementary in the repeat space theory (RST), which is the central unifying theory in the ongoing Second Generation Fukui Project.

The reader is also invited to refer to Prof. R. Hoffmann's public speech entitled 'One Culture' [27]. The present author would like to record here the fact that two Refs. [26,27] formed an important source of inspiration for the Fukui Project, which is devoted to cultivating a new interdisciplinary region in science, often utilizing dialectic interplay between a complementary pair of opposite notions and ideas. These two Refs. [26,27] are also playing a role of a guideline for the Matrix Art Program of what we call the Niagara Project (cf. [8] for details), which is a special new part of the on-going international, interdisciplinary, and inter-generational Second Generation Fukui Project.

Concluding Remarks:

(i) Multidimensional generalizations of earlier propositions and theorems concerning the Fukui conjecture are among important targets in the Second Generation Fukui Project. We remark that the family of functions $\mu_{d,n,\theta}$ defined in the present article plays a significant role in the above-mentioned generalizations.

(ii) Function $\mathrm{MagicMt}_\theta$ can be regarded as the real part of the complex-valued function $\sum_{k=0}^{\infty} (\exp(ik\theta))\mathrm{Pyramid}_k$. We remark that one can straightforwardly generalize the whole argument in Sects. 2 and 3 to complex-valued functions and complex operator algebras by considering the complex Banach spaces $C(J^d, \mathbb{C})$ and $C_0(J^d, \mathbb{C})$, and the complex Banach algebra $\boldsymbol{B}(C_0(J^d, \mathbb{C}))$. The details along these lines will be published elsewhere.

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